Solutions to Midterm II

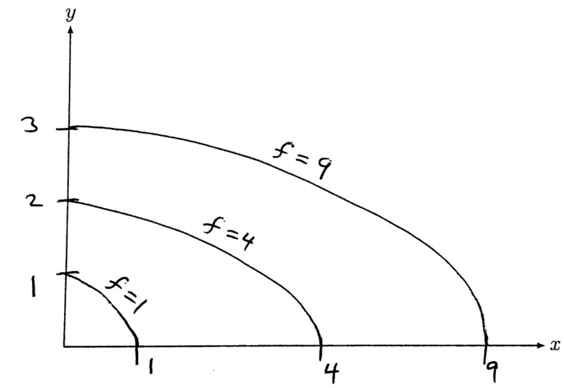
1. A level curve is

$$x + y^2 = k$$

for some value of k, equivalently

$$x = k - y^2 \quad \leftrightarrow \quad y = \sqrt{k - x}$$

Because $x \geq 0$ and $y \geq 0$, we get parts of parabolas. Values $k = 1, 4, 9$ are convenient and shown at right. (*Other values are fine too.*)



2. Integrate twice and use the initial values:

$$\mathbf{v}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{k} + \mathbf{v}(0) = (2t+3)\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k}$$

$$\mathbf{r}(t) = (t^2 + 3t)\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k} + \mathbf{r}(0)$$

$$= (t^2 + 3t)\,\mathbf{i} + (1-t)\,\mathbf{j} + \left(\frac{1}{3}t^3 + 1\right)\mathbf{k}$$

3. (a)

$$f_x = \frac{a}{ax+by}, \qquad f_y = \frac{b}{ax+by}$$

(b)

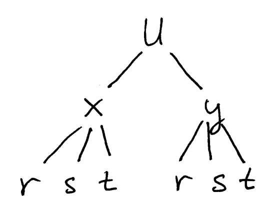
$$f_{xx} = -\frac{a^2}{(ax+by)^2}, \qquad f_{yy} = -\frac{b^2}{(ax+by)^2}, \qquad f_{xy} = f_{yx} = -\frac{ab}{(ax+by)^2}$$

4. Tree at right and chain rule below.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$



5. (a)

$$\nabla f = \left\langle y^2 \arctan(z), 2xy\arctan(z), \frac{xy^2}{1+z^2} \right\rangle$$

(b) Name the point $\mathbf{x}_0 = (2,1,1)$ for clarity. Remember to normalize the direction vector. Note $\arctan(1) = \pi/4$. Thus

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \qquad \text{and} \qquad \nabla f(\mathbf{x}_0) = \left\langle \frac{\pi}{4}, \pi, 1 \right\rangle.$$

So

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{u} = \frac{5\pi + 4}{4\sqrt{3}}$$

6. **(a)** Compute the gradient and set it to zero:

$$\nabla f = \left\langle 2x - 2y, -2x + 3y^2 - y \right\rangle$$

so solve

$$2x - 2y = 0$$
$$-2x + 3y^2 - y = 0$$

The first equation says $x = y$. The second equation now says $3y^2 - 3y = 0$ so $y = 0$ or $y = 1$. For $y = 0$ we have $x = 0$ and for $y = 1$ we have $x = 1$. Thus the two critical points are $(0, 0)$ and $(1, 1)$.

(b) For each critical point, calculate the value of $D = f_{xx}f_{yy} - f_{xy}^2 = 12y - 6$. Also note $f_{xx} = 2$. Now report results in a simple table:

critical point	sign of D	sign of f_{xx}	type
$(0, 0)$	$-$	$+$	saddle
$(1, 1)$	$+$	$+$	local minimum

7. Remember to think of z as a function of x, y. Then solve for the partial derivative:

$$e^z \frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x} \implies \frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$$
$$e^z \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y} \implies \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

8. The equation $xy^2z^3 = 8$ is a level surface of $F(x, y, z) = xy^2z^3$. Thus the equation of the plane is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

For our function the partial derivatives are

$$\nabla F = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle$$

At the point $(x_0, y_0, z_0) = (2, 2, 1)$ this is

$$\nabla F(x_0, y_0, z_0) = \langle 4, 8, 24 \rangle$$

Thus the plane is

$$4(x - 2) + 8(y - 2) + 24(z - 1) = 0$$

or, after dividing by 4,

$$x + 2y + 6z - 12 = 0$$

Extra Credit. Note $D = 0$ at a critical point if the function is locally like a cubic or higher power. Thus one answer is

- $f(x, y) = x^4 + y^4$ has a single critical point $(0, 0)$ where $D = 0$, but $f(x, y) > 0$ for $(x, y) \neq (0, 0)$, so $(0, 0)$ is a local (and absolute) minimum
- $f(x, y) = x^4 - y^4$ has a single critical point $(0, 0)$ where $D = 0$, but $f(x, 0) > 0$ for $x \neq 0$ and $f(0, y) < 0$ for $y \neq 0$, so $(0, 0)$ is a saddle point