Solutions to Midterm I

1. First compute $\mathbf{v} = \overrightarrow{PQ} = \langle 4,2,3\rangle$ and $\mathbf{w} = \overrightarrow{PR} = \langle 1,1,2\rangle$. Then

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = (4-3)\mathbf{i} - (8-3)\mathbf{j} + (4-2)\mathbf{k} = \langle 1,-5,2\rangle$$

so

$$A = \frac{1}{2}|\mathbf{v}\times\mathbf{w}| = \frac{1}{2}\sqrt{1^2+5^2+2^2} = \frac{\sqrt{30}}{2}$$

2. Since the plane is parallel to the given plane, it has normal $\mathbf{n} = \langle 5,-1,-1\rangle$. We have $\mathbf{r}_0 = \langle 1,-1,-1\rangle$ as a point on the plane. Thus an equation of the plane is

$$0 = \mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) = \langle 5,-1,-1\rangle\cdot\langle x-1,y+1,z+1\rangle = 5(x+1)-(y+1)-z(+1) = 5x-y-z-7$$

so $a=5$, $b=-1$, $c=-1$, and $d=-7$ in the desired form.

3. Divide through by 2 and complete the squares to get

$$(x-2)^2+y^2+(x+6)^2 = \frac{81}{2}$$

so the center is at $(2,0,-6)$ and the radius is $9/\sqrt{2}$.

4. At the point $(2,4)$ the slope of $y=x^2$ is $m = 2x\Big|_{x=2} = 4$. Recall $m = (\text{rise})/(\text{run})$ so $\mathbf{v} = \langle 1,4\rangle$ is a tangent vector. The two unit vectors along the tangent line are

$$\hat{\mathbf{v}}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right\rangle, \qquad \hat{\mathbf{v}}_2 = \frac{-\mathbf{v}}{|-\mathbf{v}|} = \left\langle -\frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}}\right\rangle$$

5.

$$\theta = \arccos\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \arccos\left(\frac{12-5-10}{\sqrt{16+25+4}\sqrt{9+1+25}}\right) = \arccos\left(\frac{-3}{\sqrt{45}\sqrt{35}}\right) = \arccos\left(\frac{-1}{5\sqrt{7}}\right)$$

6. Only (b), (c), and (e) are meaningful.

7. The sketch shows a circle of radius 2 centered at $(0,0,1)$ which is horizontal (i.e. in the plane $z=1$). If one looks at the curve from above then t increases in the counter-clockwise direction.

8. I chose the point in the line as $\mathbf{r}_0 = \left\langle 0,\frac{1}{2},1\right\rangle$ and the vector along the line as $\mathbf{v} = \overrightarrow{PQ} = \left\langle 2,\frac{1}{2},-4\right\rangle$. Thus one possible vector equation of the line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \left\langle 2t, \frac{1}{2}+\frac{1}{2}t, 1-4t\right\rangle$$

for all t. As parametric equations: $x(t)=2t$, $y(t) = \frac{1}{2}+\frac{1}{2}t$, $z(t) = 1-4t$. There are other correct answers.

Extra Credit. From the formula for the arclength function it follows that $ds/dt = |\mathbf{r}'(t)|$. Also recall $\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)|$ and $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$. Thus

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} = \frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \kappa\mathbf{N}$$

9. **(a)** Note $\mathbf{r}'(t) = \left\langle 0,t^2,t\right\rangle$ so, using $u = t^2+1$,

$$L = \int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 \sqrt{t^4+t^2}\,dt = \int_0^2 t\sqrt{t^2+1}\,dt = \frac{1}{2}\int_1^5 u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big]_1^5 = \frac{1}{3}(5\sqrt{5}-1)$$

(b)

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\left\langle 0,t^2,t\right\rangle}{\sqrt{t^4+t^2}} = \left\langle 0, t(t^2+1)^{-1/2}, (t^2+1)^{-1/2}\right\rangle$$

(c) From **(b)** it follows that

$$\mathbf{T}'(t) = \left\langle 0, (t^2+1)^{-1/2} + t(-1/2)(t^2+1)^{-3/2}(2t), (-1/2)(t^2+1)^{-3/2}(2t)\right\rangle = (t^2+1)^{-3/2}\left\langle 0,1,-t\right\rangle$$

so

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{(t^2+1)^{-3/2}\sqrt{t^2+1}}{t\sqrt{t^2+1}} = \frac{1}{t(t^2+1)^{3/2}}$$