Solutions to Final Exam

1. VERSION A: IV, III, I, II; VERSION B: IV, I, II, III

2. VERSION A: Here $P(x,y) = x^2 + y$, $Q(x,y) = 3x - y^2$ so by Green's theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 3 - 1\,dA = 2\iint_D dA = 2A(D) = 12$$

VERSION B: Everything the same except $A(D) = 5$ so the final answer is 10.

3. (*Many people missed the "ds".*) We parameterize the line,

$$\mathbf{r}(t) = (1-t)\langle 2,0\rangle + t\langle 5,4\rangle = \langle 2+3t, 4t\rangle, \qquad 0 \le t \le 1$$

Thus $ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt = \sqrt{3^2+4^2}\,dt = 5\,dt$ and

$$\int_C xe^y\,ds = \int_0^1 (2+3t)e^{4t}5\,dt = 5\left(\left[(2+3t)\frac{1}{4}e^{4t}\right]_0^1 - \int_0^1 \frac{3}{4}e^{4t}\,dt\right) = \frac{5}{16}\left(17e^4 - 5\right)$$

4. If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is conservative ($\mathbf{F} = \nabla f$) then $P = f_x$ and $Q = f_y$ for some function $f(x,y)$. This is possible because these are equal:

$$\frac{\partial Q}{\partial x} = e^x + \cos y = \frac{\partial P}{\partial y}$$

That is, $\mathbf{F}$ is conservative. Now we find $f(x,y)$ by taking the x-antiderivative of $f_x = ye^x + \sin y$:

$$f(x,y) = ye^x + x\sin y + g(y)$$

Now match the y derivatives:

$$e^x + x\cos y + g'(y) = f_y = Q = e^x + x\cos y$$

so $g'(y) = 0$ so $g(y) = C$. Since we want only *an* antiderivative we may choose $C = 0$ so finally

$$f(x,y) = ye^x + x\sin y$$

5. (*A sketch of the tetrahedron is essential. Setting up limits of integration gave the most trouble.*) Here is one of 6 different orders of integration:

$$m = \iiint_E \rho\,dz\,dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} 2x\,dz\,dy\,dx = \int_0^1\int_0^{1-x} 2x(1-x-y)\,dy\,dx$$

$$= \int_0^1 2x\left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 x(1-x)^2\,dx = \int_0^1 x - 2x^2 + x^3\,dx = \left[\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_0^1 = \frac{1}{12}$$

6. Use $u = x^2$ in the integral over x:

$$\iint_D x\cos y\,dA = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2} x\cos y\,dy\,dx = \int_0^1 x\sin(x^2)\,dx = \frac{1}{2}\int_0^1 \sin u\,du = \frac{1}{2}\left(1 - \cos(1)\right)$$

7. (a) The sketch shows something like an ice cream cone. Note that the cone $z = \sqrt{x^2+y^2}$ corresponds to the surface which is angle $\pi/4$ from the z-axis, i.e. it is $\phi = \pi/4$.

(b)

$$V = \iiint_E 1\,dV = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\int_{\rho=0}^{\rho=1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\left(\int_0^{\pi/4}\sin\phi\,d\phi\right)\left(\int_0^1 \rho^2\,d\rho\right)$$

$$= 2\pi\left(1 - \cos(\pi/4)\right)\frac{1}{3} = \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

8. Use the cross product to get a vector which is orthogonal to both given vectors. Then normalize it:

$$\mathbf{v} = (\mathbf{i}+\mathbf{j})\times(\mathbf{i}+\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 0\\ 1 & 0 & 1\end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

so

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} - \mathbf{j} - \mathbf{k}}{\sqrt{1+1+1}} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

9. The normal vector of the given plane is $\mathbf{n} = \langle 5, -1, -1 \rangle$. The points $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle 1, -1, -1 \rangle$ should be on the plane. Thus the equation is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ or equivalently

$$5(x-1) + (-1)(y-(-1)) + (-1)(z-(-1)) = 0$$

or, in the requested form, $5x - y - z - 7 = 0$.

10.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_1^5 \sqrt{(2t)^2 + (3t^2)^2 + (4t^3)^2}\, dt = \int_1^5 t\sqrt{4 + 9t^2 + 16t^4}\, dt$$

11. **(a)** The curve is a quarter of an ellipse, starting at $(0, 2)$ and ending at $(4, 0)$.

(b)

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 4\cos t, -2\sin t \rangle}{\sqrt{(4\cos t)^2 + (-2\sin t)^2}} = \frac{\langle 2\cos t, -\sin t \rangle}{\sqrt{4\cos^2 t + \sin^2 t}}$$

12. Use the partial derivatives $f_x = y/(2\sqrt{xy})$, $f_y = x/(2\sqrt{xy})$ and the point $(x_0, y_0) = (1, 4)$ in the linearization formula:

$$\begin{aligned} L(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &= \sqrt{1\cdot 4} + \frac{4}{2\sqrt{1\cdot 4}}(x-1) + \frac{1}{2\sqrt{1\cdot 4}}(y-4) = 2 + (x-1) + \frac{1}{4}(y-4) = x + \frac{1}{4}y \end{aligned}$$

13. Apply the chain rule, reduce to variables r, θ only, and then fill in the values:

$$\begin{aligned} \frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)r \\ &= (r\sin\theta + r\theta)(-r\sin\theta) + (r\cos\theta + r\theta)(r\cos\theta) + (r\sin\theta + r\cos\theta)r \\ &= r^2\left(-(\sin\theta + \theta)\sin\theta + (\cos\theta + \theta)\cos\theta + (\sin\theta + \cos\theta)\right) \\ &= 2^2\left(-(1 + \frac{\pi}{2})1 + (0 + \frac{\pi}{2})0 + (1+0)\right) = 4\left(-1 - \frac{\pi}{2} + 1\right) = -2\pi \end{aligned}$$

14. **(a)** Compute the gradient: $\nabla f = \langle -4x^3 + 4x, -2y \rangle$. Set it to zero and solve two equations:

$$x^3 - x = x(x-1)(x+1) = 0, \qquad y = 0$$

There are three solutions, i.e. critical points: $(-1, 0)$, $(0, 0)$, $(+1, 0)$

(b) To determine the type of the critical points we write down

$$D = f_{xx}f_{yy} - f_{xy}^2 = \left(-12x^2 + 4\right)(-2) - 0^2 = 24x^2 - 8$$

and $f_{xx} = -12x^2 - 4$. Then:

- $(-1, 0)$: $D > 0$ and $f_{xx} < 0$ so <u>local maximum</u>
- $(0, 0)$: $D < 0$ so <u>saddle point</u>
- $(1, 0)$: $D > 0$ and $f_{xx} < 0$ so <u>local maximum</u>

Extra Credit. The given formula defines a function $f(x, y)$ because the integral is path-independent because $\mathbf{F}$ is conservative. Also, because $\mathbf{F}$ is conservative, there is *some* function $g(x, y)$ so that $\mathbf{F} = \nabla g$. Then by the Fundamental Theorem for line integrals we have

$$\int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r} = \int_{(a,b)}^{(x,y)} \nabla g \cdot d\mathbf{r} = g(x,y) - g(a,b)$$

So let $f(x, y) = g(x, y) - g(a, b)$. Then $\nabla f = \nabla g = \mathbf{F}$.

Sources of the problems. **1** = §16.1 # 11–14; **2** = §16.4 # 28; **3** = §16.2 # 4; **4** = §16.3 # 7; **5** = §15.6 # 42; **6** = §15.2 # 17; **7** = §15.3 # 25; **8** = §12.3 # 27; **9** = §12.5 # 27; **10** = §13.3 # 7; **11** = §13.2 # 7(a); **12** = §14.4 # 12; **13** = §14.5 # 23; **14** = §14.7 # 10