Math 252
Exam 4

Key

Name

"I used to go away for weeks in a state of confusion."
--Albert Einstein

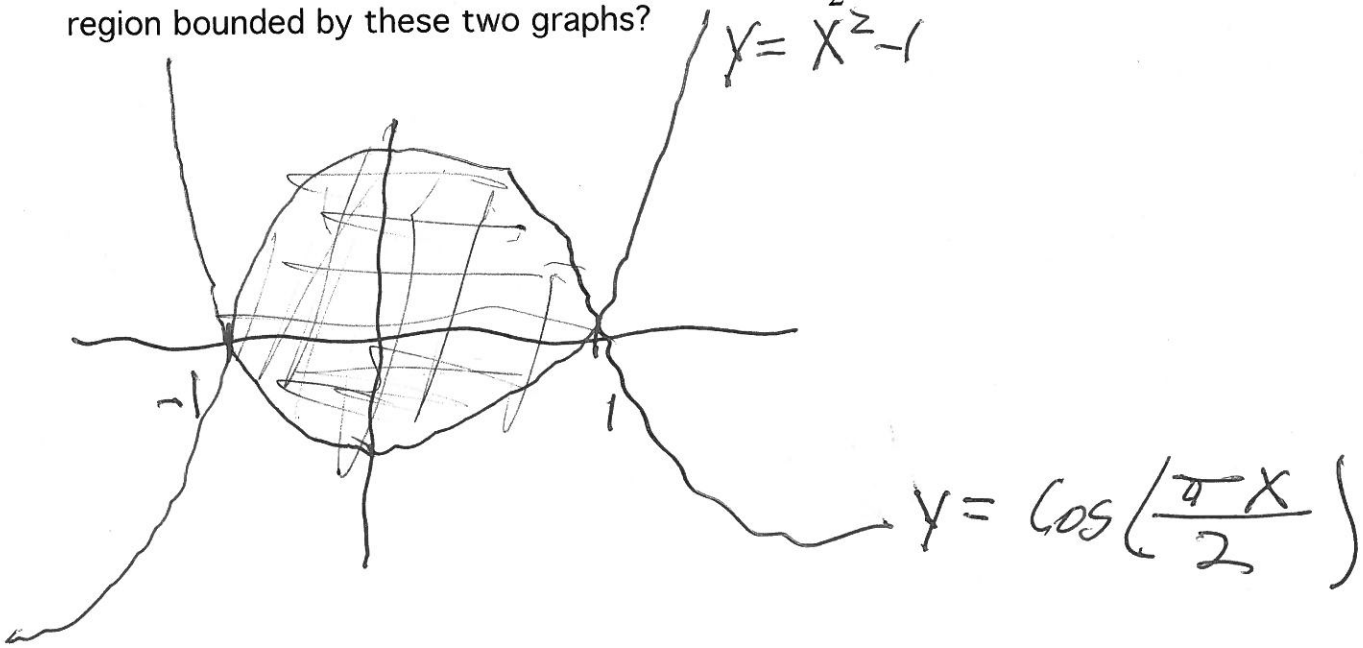
EACH SECTION OF THIS EXAM IS WORTH 10 POINTS.

1. Suppose f(x)= Sin(x). Compute the average value of f on [ 0, $\pi$ ].

$$\frac{\int_0^{\pi} \sin(x)\,dx}{\pi - 0} = \frac{-\cos(x)\Big|_0^{\pi}}{\pi}$$

$$= \frac{2}{\pi}$$

2. Consider the graphs of $y = x^2 - 1$ and $y=\text{Cos}(\frac{\pi x}{2})$. What is the area of the region bounded by these two graphs?




$$\int_{-1}^{1} \left[\text{Cos}\left(\frac{\pi x}{2}\right) - (x^2-1)\right] dx$$

$$= \frac{\text{Sin}\left(\frac{\pi x}{2}\right)}{\frac{\pi}{2}} - \frac{x^3}{3} + 1x \Bigg|_{-1}^{1}$$

$$= \frac{4}{\pi}\text{Sin}\left(\frac{\pi}{2}\right) - \frac{2}{3} + 2$$

$$= \frac{4}{\pi} + \frac{4}{3}$$

3. A particular chain weighs one pound for every four feet. It is attached to a bucket of coal that weighs 40 pounds. If the bucket is pulled up a mine shaft that is 100 feet deep, how much work is done?

$$\int_0^{100} \left[40 + \frac{1}{4}x\right] dx$$

$$= 40x + \frac{1}{4}\frac{x^2}{2}\Big|_0^{100}$$

$$= 4000 + \frac{1}{8}10000$$

$$= 4000 + 1250$$

$$= 5250 \text{ Ft-Lbs}$$

4. A tank of jet fuel is 10 feet tall. Any horizontal cross-section of the tank is a rectangle, four feet wide and six feet long. The jet fuel weighs 50 pounds per cubic foot. A pump empties the full tank by lifting the fuel three feet above the tank and then discarding it. How much work is done?

$$\int_0^{10} 50\,[x+3]\,4\cdot 6\,\partial x$$

$$= 1200\left[\frac{x^2}{2} + 3x\right]\Big|_0^{10}$$

$$= 1200\,[50+30]$$

$$= 96000 \text{ Ft-Lbs}$$

5. Find the area enclosed by the graph of $r = 3+\sin(\theta)$.

$$\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (3+\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} [9 + 6\sin\theta + \sin^2\theta] \, d\theta$$

$$= \frac{1}{2} \left[ 9\theta - 6\cos\theta \Big|_0^{2\pi} + \int_0^{2\pi} \frac{1-\cos 2\theta}{2} \, d\theta \right]$$

$$= \frac{1}{2} \left[ 18\pi - 0 + \left( \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) \Big|_0^{2\pi} \right]$$

$$= \frac{19\pi}{2}$$

6. Determine if the following converge or diverge:

(a) $\sum_{k=0}^{\infty} \frac{10k}{2+100k}$ Diverges by Divergence Test

$$\frac{10k}{2+100k} = \frac{10}{\frac{2}{k}+100} \to \frac{10}{0+100}$$

$$= \frac{1}{10} \neq 0$$

(b) $\sum_{k=1}^{\infty} \frac{k^4}{2+k^6}$

Converges. Comparison test with $\sum \frac{1}{k^2}$, a Convergent p-series.

Note, $0 < \frac{k^4}{2+k^6} \leq \frac{k^4}{k^6} = \frac{1}{k^2}$

7. Suppose $y=\int_2^x \sqrt{t^4-1}\,dt$. Find the arc length on the interval $4\le x\le 6$.

$$y' = \sqrt{x^4-1}$$

$$\int_4^6 \sqrt{1+x^4-1}\,dx$$

$$= \int_4^6 x^2\,dx = \frac{x^3}{3}\Big|_4^6$$

$$= \frac{216}{3} - \frac{64}{3} = \frac{152}{3}$$

8. Consider the parameterized expression $x = 2 + 3\text{Cos}(t)$ and $y = 2 + 3\text{Sin}(t)$ where $-1 \le t \le 1$. Find the length of the graph.

$$x' = -3\sin t$$

$$y' = 3\cos t$$

$$\int_{-1}^{1} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_{-1}^{1} 3\, dt = 3t \Big|_{-1}^{1} = 6$$

9. A swimming pool is 10 feet deep. One wall on the side of the pool is rectangular in shape. It is 20 feet wide. Find the force on that side of the pool if it is filled with a liquid having a density of 60 pounds per cubic foot.

$$\int_0^{10} 60 \cdot 20\, x \, dx$$

$$= 1200 \left. \frac{x^2}{2} \right|_0^{10}$$

$$= 60{,}000 \text{ Lbs}$$

10. Express as a closed form expression: $\sum_{k=0}^{\infty} \frac{x^{2k+3}}{4^k}$ .

$$= x^3 \sum_{K=0}^{\infty} \left(\frac{x^2}{4}\right)^K$$

$$= x^3 \frac{1}{1 - \frac{x^2}{4}}$$

$$= \frac{4x^3}{4 - x^2}$$