Your Name

Instructor Name (Zirbes)

Start Time

End Time

Page	Total Points	Score
2	20	
3	14	
4	14	
5	16	
6	18	
7	18	
8	(5 Extra Credit)	
Total	100	

- You will have 1 hour to complete the test.
- This test is closed notes and closed book and you may not use a calculator.
- Label any diagrams so as to indicate axes labels and scale.
- In order to receive full credit, you must **show your work.** Please write out your computations on the exam paper.
- Clearly explain your answers using complete sentences when applicable. Cite any relevant tests or theorems by name.
- **PLACE A BOX AROUND YOUR FINAL ANSWER to each question** where appropriate.

1. (12 points) Let $a_n = \dfrac{2n+1}{5n+3}$ for $n \geq 1$.

   (a) List the first two terms in the sequence $\{a_n\}$.

   $a_1 = \frac{3}{8}$

   $a_2 = \frac{5}{13}$

   (b) Determine whether the sequence $\{a_n\}$ converges.

   $\lim_{n\to\infty} a_n = 2/5$, so a_n conv. to $2/5$

   (c) Let $S = \sum_{n=1}^{\infty} a_n$. Calculate s_1, and s_2, the first two terms of the sequence of partial sums.

   $s_1 = a_1 = \boxed{3/8}$

   $s_2 = a_1 + a_2 = \frac{3}{8}\left(\frac{13}{13}\right) + \frac{5}{13}\left(\frac{8}{8}\right) = \frac{39+40}{104} = \boxed{\frac{79}{104}}$

   (d) Does the series $S = \sum_{n=1}^{\infty} a_n$ converge or diverge? Explain!

   As $\lim_{n\to\infty} a_n = 2/5 \neq 0$, the series diverges by the test for divergence.

2. (8 points) Determine whether the series is conditionally convergent, absolutely convergent, or divergent. Explain why in each case.

   (a) $\sum_{n=1}^{\infty} (-1)^n n^{-9/10} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{9/10}}$

   ① The series converges by AST.

   ② $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{9/10}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{9/10}}$, is a divergent p-series.

   Thus, the series is conditionally convergent.

   (b) $\sum_{n=1}^{\infty} (-1)^n n^{-3/2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$

   ① The series conv. by AST

   ② $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{3/2}} \right| = \sum_{n=1}^{\infty} 1/n^{3/2}$, which is a conv. p-series.

   The series is abs. conv.

3. (14 points) Determine if the series below converge or diverge. Full credit will only be given for answers that include (1 pt) the name of the test being applied, (5 pts) a complete application of the test, including evidence that the conditions have been met, and (1 pt) a clear conclusion with justification.

(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{4n^6+2}}$

LCT w/ $a_n = \frac{n^2+1}{\sqrt{4n^6+2}}$, $b_n = \frac{n^2}{\sqrt{n^6}} = \frac{n^2}{n^3} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+1}{\sqrt{4n^6+1}} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^3+n}{\sqrt{4n^6+1}}$$

$$= \lim_{n\to\infty} \frac{1+1/n^2}{\sqrt{4+1/n^6}}$$

$$= 1/2$$

As $\sum a_n$ div ($p=1$ or harmonic), $\sum b_n$ also div.

(b) $\sum_{n=1}^{\infty} \frac{\sin(3n)}{5+3^n}$

Note $0 < \left| \frac{\sin 3n}{5+3^n} \right| < \frac{1}{3^n}$.

As $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a geometric series w/ $r = 1/3$ it is convergent.

Thus $\sum_{n=1}^{\infty} \left| \frac{\sin 3n}{5+3^n} \right|$ conv. by DCT.

This means the original series is absolutely convergent, which implies the original series is convergent.

4. (14 points) Determine if the series below converge or diverge. Full credit will only be given for answers that include (1 pt) the name of the test being applied, (5 pts) a complete application of the test, including evidence that the conditions have been met, and (1 pt) a clear conclusion with justification.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+9}$ AST, Let $b_n = \frac{n}{n^2+9}$

① $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+9}$

$= \lim_{n\to\infty} \frac{1}{n+9/n}$

$=0$

② To show b_n is dec. we use $f'(x)$.

$f(x) = \frac{x}{x^2+9}$, $f'(x) = \frac{(x^2+9) - x\cdot 2x}{(x^2+9)^2}$

$= \frac{9-x^2}{(x^2+9)^2} < 0$ if $x > 3$.

Thus b_n is dec.

③ The series conv. by AST.

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ Int test

① Let $f(x) = \frac{1}{x(\ln x)^5}$.

② Note $f(x)$ is cts + pos on $[2,\infty)$, also $f(x)$ is obviously decreasing.

③ Now $\int_2^{\infty} \frac{1}{x(\ln x)^5}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^5}dx$

$u = \ln x$
$du = \frac{1}{x}dx$
$x=2, u=\ln 2$
$x=b, u=\ln b$

$= \lim_{b\to\infty}\int_{\ln 2}^{\ln b} u^{-5}\,dx$

$= \lim_{b\to\infty} \frac{u^{-4}}{-4}\Big|_{\ln 2}^{\ln b}$

$= \lim_{b\to\infty}\left(\frac{-1}{4(\ln b)^4} + \frac{1}{4(\ln 2)^2}\right)$

$= \frac{1}{4(\ln 2)^2}$

④ As the improper integral converges, so does the series.

5. (16 points) Find the radius and interval of convergence of the following series. If applicable, clearly explain why the series does or does not converge at the endpoints.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(2n+3)!} \quad \frac{(2n+1)!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{(2n+3)(2n+2)}$$

$$= 0 < 1$$

so $\boxed{R = \infty}$

$\boxed{\text{IOC } (-\infty, \infty)}$

b) $\displaystyle\sum_{n=0}^{\infty} n^n (x+3)^n$

Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| n^n (x+3)^n \right|}$$

$$= \lim_{n\to\infty} n \, |x+3|$$

$$= \infty$$

$\boxed{\text{Thus } R = 0 \quad \text{IOC } \{-3\}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-6)^n}{\sqrt{n}\,4^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-6)^{n+1}}{\sqrt{n+1}\; 4^{n+1}} \cdot \frac{4^n \sqrt{n}}{(x-6)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x-6|}{4} \frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= \frac{|x-6|}{4} \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$

$$= \frac{|x-6|}{4}$$

conv. if $\dfrac{|x-6|}{4} < 1$

$\Rightarrow |x-6| < 4$

$\Rightarrow -4 < x-6 < 4$

$\Rightarrow 2 < x < 10$

@ $x=2$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{\sqrt{n}\, 4^n}$

$= \displaystyle\sum_{n=1}^{\infty} 1/\sqrt{n}$ $p = 1/2$, div.

@ $x=10$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}\, 4^n}$

$= \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv. AST

$\boxed{\text{IOC } (2, 10]}$

$\boxed{R = 4}$

6. (18 points) Find the sum of the following series exactly. If the series diverges explain why.

<u>Geo</u>

(a) $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5 \cdot 11^n}$ $= \sum_{n=0}^{\infty} \frac{2^{2(n+1)+1}}{5 \cdot 11^{n+1}}$ $= \boxed{8/35}$

$$= \sum_{n=0}^{\infty} \frac{2^3 \, 4^n}{55 \cdot 11^n}$$

$$= \sum_{n=0}^{\infty} \frac{8}{55} \left(\frac{4}{11}\right)^n$$

$$= \frac{8/55 \cdot 55}{(1 - 4/11)\,55}$$

$$= \frac{8}{55 - 20}$$

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{9^n n!}$ $= \sum_{n=0}^{\infty} \frac{(-2/9)^n}{n!}$

Note: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$= \boxed{e^{-2/9}}$$

<u>Telescoping</u>

(c) $\sum_{n=1}^{\infty} \left(e^{1/(n+2)} - e^{1/n}\right)$

$$S_n = (e^{1/3} - e) + (e^{1/4} - e^{1/2}) + (e^{1/5} - e^{1/3}) + \cdots$$

$$\cdots + (e^{1/n} - e^{1/(n-2)}) + (e^{1/(n+1)} - e^{1/(n-1)}) + (e^{1/(n+2)} - e^{1/n})$$

$$= -e - e^{1/2} + e^{1/(n+1)} + e^{1/(n+2)}$$

$$S = \lim_{n \to \infty} \left(-e - \sqrt{e} + e^{1/(n+1)} + e^{1/(n+2)}\right)$$

$$= \boxed{2 - e - \sqrt{e}}$$

7. (10 points) Given $f(x) = \dfrac{x^5}{x^3+27}$.

(a) Find a power series for $f(x)$ using a geometric series and give the radius of convergence.

$$f(x) = \frac{x^5/27}{1-(-x^3/27)}$$

$$= \sum_{n=0}^{\infty} \frac{x^5}{27}\left(\frac{-x^3}{27}\right)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+5}}{27^{n+1}}}$$

conv. if $\left|-x^3/27\right| < 1$

OR $|x^3| < 27$

OR $\boxed{|x| < 3}$

(b) Using your result from (a), evaluate $\int f(x)dx$ as a power series. State the radius of convergence.

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+5}}{27^{n+1}}\,dx$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+6}}{27^{n+1}(3n+6)} + C}$$

$\boxed{\text{for } |x| < 3}$

8. (8 points) Find the Taylor Series for $f(x) = \dfrac{1}{x}$ centered at $a = 2$ using the definition. You must write your answer using summation notation. Simplify and cancel if applicable.

$f(x) = 1/x = x^{-1}$

$f'(x) = -1x^{-2} = -1/x^2$

$f''(x) = 2x^{-3} = 2/x^3$

$f'''(x) = -6x^{-4} = -6/x^4$

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

$$= \frac{1}{2} - \frac{1}{2^2}(x-2) + \frac{2}{2^3}\frac{1}{2!}(x-2)^2 - \frac{6}{2^4}\frac{1}{3!}(x-2)^3 + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \frac{(x-2)^3}{2^4} + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{2^{n+1}}}$$

9. (5 points extra credit) Taylor's Inequality, states that if $|f^{(n+1)}(x)| \leq M$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

$\overset{2}{6}4$
$\underline{\;\;6}$
$\;\;4$

(a) (2 points) Given $f(x) = x^{3/2}$ at $a = 1$ find the 3rd degree Taylor polynomial $T_3(x)$.

$f(x) = x^{3/2}$

$f'(x) = \frac{3}{2}x^{1/2}$

$f''(x) = \frac{3}{4}x^{-1/2}$

$f'''(x) = \frac{-3}{8}x^{-3/2}$

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$
$$= 1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 + \frac{1}{3!}\left(\frac{-3}{8}\right)(x-1)^3$$
$$= \boxed{1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{1}{16}(x-1)^3}$$

$-3/8 \cdot 1/8$

(b) (3 points) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when x is in the interval $[+0.5, 1.5]$

Note $f''''(x) = \frac{9}{16}x^{-5/2}$.

on $[0,1]$, $|f''''(x)| = \left|\frac{9}{16x^{5/2}}\right|$ is largest at $x = 0.5$

so $|f''''(x)| \leq \frac{9}{16(0.5)^{5/2}} = M$, and $0.5 \leq x \leq 1.5$

$\Rightarrow -0.5 \leq x-1 \leq 0.5$

$\Rightarrow |x-1| < 0.5$

$$\boxed{|R_3(x)| \leq \frac{9}{16(0.5)^{5/2}} \cdot \frac{1}{4!}(0.5)^4}$$