Your Name

Key

Instructor

Start Time

End Time

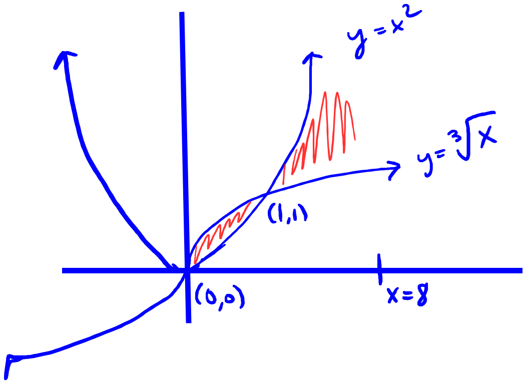
Page	Total Points	Score
2	10	
3	15	
4	20	
5	15	
6	25	
7	15	
8	(5 Extra Credit)	
Total	100	

- You will have 1 hour to complete the test.
- This test is closed notes and closed book and you may not use a calculator.
- Label any diagrams so as to indicate axes labels and scale.
- In order to receive full credit, you must **show your work.** Please write out your computations on the exam paper.
- Carefully read all problems as some problems are set-up only.
- **PLACE A BOX AROUND $\boxed{\textbf{YOUR FINAL ANSWER}}$ to each question** where appropriate.

1. (10 points) Consider the region bounded by $y = x^2$ and $y = \sqrt[3]{x}$ for $0 \le x \le 8$.

(a) (2 points) Sketch the region bounded by the curves. Clearly label each curve and notate any important points.

label



(b) (8 points) Find the area of this region.

$$A = \int_0^1 \left(\sqrt[3]{x} - x^2\right) dx + \int_1^8 \left(x^2 - \sqrt[3]{x}\right) dx$$

$$= \left[\frac{3}{4}x^{4/3} - \frac{1}{3}x^3\right]_0^1 + \left[\frac{1}{3}x^3 - \frac{3}{4}x^{4/3}\right]_1^8$$

$$= \left[\left(\frac{3}{4} - \frac{1}{3}\right) - (0)\right] + \left[\left(\frac{1}{3}(8)^3 - \frac{3}{4}(8)^{4/3}\right) - \left(\frac{1}{3} - \frac{3}{4}\right)\right]$$

$$= \frac{5}{12} + \left(\frac{512}{3} - \frac{48}{4}\right) + \frac{5}{12}$$

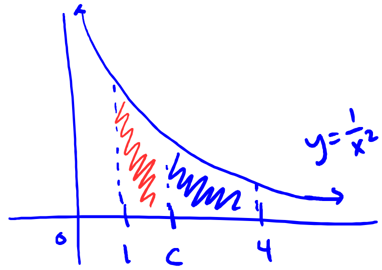
$$= \frac{10}{12} + \frac{2048}{12} - \frac{144}{12}$$

$$= \frac{1914}{12}$$

$$= \boxed{\frac{319}{2}}$$

2. (8 points) In this problem, you are going to find the number a such that the line $x = a$ divdes the area under the curve $y = 1/x^2$ into two regions of eqal area for $1 \le x \le 4$.

   (a) (2 points) Sketch a rough graph of the region and clearly indicate what you are looking for.



   (b) (6 points) Now, find the number a such that the line $x = a$ divdes the area under the curve $y = 1/x^2$ into two regions of eqal area for $1 \le x \le 4$.

$$\int_1^a \frac{1}{x^2}\,dx = \int_a^4 \frac{1}{x^2}\,dx$$

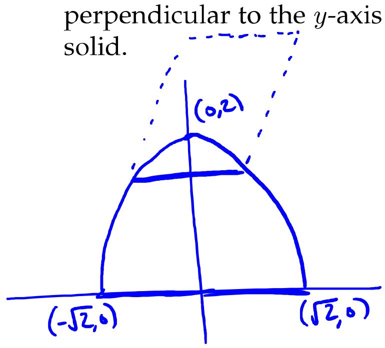
$$\Rightarrow \left[-\frac{1}{x}\right]_1^a = \left[-\frac{1}{x}\right]_a^4$$

$$\Rightarrow 1 - \frac{1}{a} = \frac{1}{a} - \frac{1}{4}$$

$$\Rightarrow \frac{2}{a} = \frac{5}{4}$$

$$\Rightarrow a = \frac{8}{5}.$$

3. (7 points) The base of a solid is the region bounded by $y = 2 - x^2$ and the x-axis. Cross sections perpendicular to the y-axis are rectangles whose height is twice the length. Find the volume of this solid.



$y = 2 - x^2$

$\Rightarrow x^2 = 2 - y$

$\Rightarrow x = \pm\sqrt{2-y}$

$\Rightarrow$ Length : $2\sqrt{2-y}$

$\Rightarrow$ Height : $4\sqrt{2-y}$

$$V = \int_0^2 \left(2\sqrt{2-y}\right)\left(4\sqrt{2-y}\right)dy$$

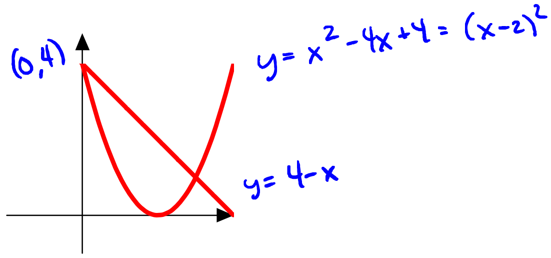
$$= 8\int_0^2 (2-y)\,dy$$

$$= 8\left[2y - \frac{1}{2}y^2\right]_0^2$$

$$= 8\left[(4-2) - (0)\right]$$

$$= 16.$$

4. (20 points, 5 points each) Consider the region bounded by $y = x^2 - 4x + 4$, $y = 4 - x$, which is graphed below. Set up, but do not solve, an integral that finds the volume of this region when it is rotated about each of the following axes. **You do NOT need to simplify the integrand.** State the method that you are using. A sketch of the region has been provided for you.



$x^2 - 4x + 4 = 4 - x$

$\Rightarrow x^2 - 3x = 0$

$\Rightarrow x(x-3) = 0$

$\Rightarrow x = 0, x = 3$

(a) x-axis Disc.

$$V = \pi \int_0^3 \left[(4-x)^2 - (x-2)^4\right] dx$$

$$= \pi \int_0^3 \left(-x^4 + 8x^3 - 23x^2 + 24x\right) dx$$

(b) y-axis Shell.

$r(x) = x$

$h(x) = (4-x) - (x^2 - 4x + 4)$

$= -x^2 + 3x$

$$V = 2\pi \int_0^3 x(-x^2 + 3x)\, dx$$

$$= 2\pi \int_0^3 (-x^3 + 3x^2)\, dx.$$

(c) $x = -3$ Shell.

$r(x) = x + 3$

$h(x) = -x^2 + 3x$

$$V = 2\pi \int_0^3 (x+3)(-x^2 + 3x)\, dx$$

$$= 2\pi \int_0^3 (-x^3 + 9x)\, dx.$$

(d) $y = 5$ Disc.

$r_{out} = 5 - (x-2)^2$

$r_{in} = 5 - (4 - x)$

$$V = \pi \int_0^2 \left[\left(5 - (x-2)^2\right)^2 - \left(5 - (4-x)\right)^2\right] dx$$

$$= \pi \int_0^2 \left[(-x^2 + 4x + 1)^2 - (1+x)^2\right] dx$$

$$= \pi \int_0^2 (x^4 - 8x^3 + 13x^2 + 6x)\, dx.$$

5. (15 points) The temperature in a certain city (in °F) t hours after 9 AM was modeled by the function

$$T(t) = 5\sin\left(\frac{\pi t}{12}\right) - 20.$$

(a) (8 points) Use this equation to find the average temperature during the period from 8 AM to 8 PM. Give an exact answer with proper units AND then, using $\pi \approx 3$ give a rough estimate of what the average temperature is to the nearest whole number.

$$T_{ave} = \frac{1}{12}\int_0^{12}\left(5\sin\left(\frac{\pi t}{12}\right) - 20\right)dt$$

$$= \frac{1}{12}\left[-\left(\frac{12}{\pi}\right)5\cos\left(\frac{\pi t}{12}\right) - 20t\right]_0^{12}$$

$$= \frac{1}{12}\left[\left(-\frac{60}{\pi}(-1) - 20(12)\right) - \left(-\frac{60}{\pi}(1) - 0\right)\right]$$

$$= \frac{1}{12}\left[\frac{120}{\pi} - 20(12)\right]$$

$$= \boxed{\left(\frac{10}{\pi} - 20\right)\ {}^\circ F}$$

$$\approx \left(\frac{10}{3} - 20\right)$$

$$= \frac{10-60}{3} = -\frac{50}{3} = -16.\bar{6} \approx \boxed{-17\ {}^\circ F}$$

(b) (2 points) Explain why the Mean Value Theorem for integrals applies to the equation $T(t)$ on any interval $[a, b]$.

Mean value theorem applies because $T(t)$ is continuous on any interval $[a,b]$.

(c) (5 points) Find the time $t = c$ such that $T(c)$ is equal to the average value from part (a). Use the exact, not the approximate answer to do this. Give units with your answer.

$$T(c) = \frac{10}{\pi} - 20$$

$$\Rightarrow 5\sin\left(\frac{\pi c}{12}\right) - 20 = \frac{10}{\pi} - 20$$

$$\Rightarrow 5\sin\left(\frac{\pi c}{12}\right) = \frac{10}{\pi}$$

$$\Rightarrow \sin\left(\frac{\pi c}{12}\right) = \frac{2}{\pi}$$

$$\Rightarrow \frac{\pi c}{12} = \sin^{-1}\left(\frac{2}{\pi}\right)$$

$$\Rightarrow c = \frac{12}{\pi}\sin^{-1}\left(\frac{2}{\pi}\right) \text{ hours after 8am.}$$

6. (8 points) If the work required to stretch a spring 2 feet beyond its natural length is 24 ft-lbs, how much work is needed it stretch it one foot beyond its natural length? Give your final answer with proper units.

$$24 = \int_0^2 kx\,dx$$
$$\Rightarrow 24 = \frac{1}{2}kx^2\Big|_0^2$$
$$\Rightarrow 24 = 2k$$
$$\Rightarrow k = 12.$$

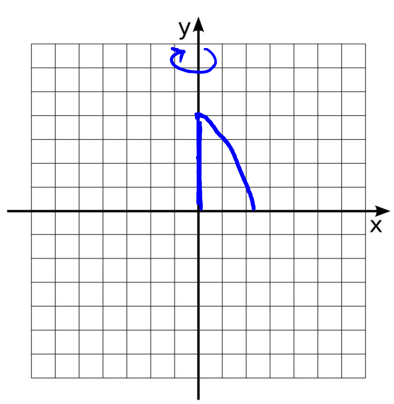
$$W = \int_0^1 12x\,dx$$
$$= 6x^2\Big|_0^1$$
$$= 6 \text{ ft-lbs.}$$

7. (7 points) Find the exact length of the curve $y = 1 + 6x^{3/2}$ for $0 \le x \le 1$.

$$y' = 6\left(\frac{3}{2}\right)x^{1/2}$$
$$= 9x^{1/2}.$$

$$L = \int_a^b \sqrt{1 + (y')^2}\,dx$$
$$= \int_0^1 \sqrt{1 + 81x}\,dx$$
$$= \frac{2}{3}\left(\frac{1}{81}\right)(1 + 81x)^{3/2}\Big|_0^1$$
$$= \frac{2}{243}\left(82^{3/2} - 1\right).$$

8. (8 points) Find the surface area obtained by rotating the region $y = 4 - x^2$ between $x = 0$ and $x = 2$ about the y-axis. Begin by sketching the curve.

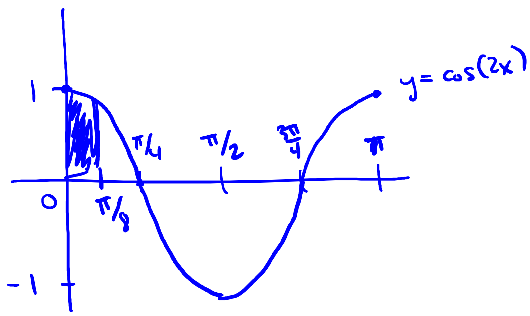


$$y' = -2x$$

$$SA = \int 2\pi x\,dS$$
$$= 2\pi \int_0^2 x\sqrt{1 + 4x^2}\,dx$$
$$= 2\pi\left(\frac{2}{3}\right)\left(\frac{1}{8}\right)\left(1 + 4x^2\right)^{3/2}\Big|_0^2$$
$$= \frac{\pi}{6}\left(17^{3/2} - 1\right).$$

9. (15 points) Consider the region bounded by $y = \cos(2x)$, $x = 0$, and $y = 0$ on the interval $[0, \pi/8]$.

(a) (5 points) Sketch curves on the interval from $[0, \pi]$, shade the region, and then find the area bounded by the curves.



$$A = \int_0^{\pi/8} \cos(2x)\,dx$$
$$= \frac{1}{2}\sin(2x)\Big|_0^{\pi/8}$$
$$= \frac{1}{2}\left[\sin\left(\frac{\pi}{4}\right) - \sin(0)\right]$$
$$= \frac{\sqrt{2}}{4}.$$

(b) (10 points) Find the centroid (also known as the center of mass) of this region.

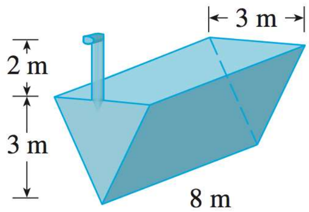
$u = x$ $\quad v = \frac{1}{2}\sin(2x)$

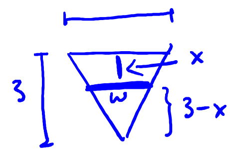
$du = dx$ $\quad dv = \cos(2x)dx$

$$\bar{x} = \frac{1}{A}\int_0^{\pi/8} x\cos(2x)\,dx$$
$$= \frac{1}{A}\left[\frac{1}{2}x\sin(2x) - \int_0^{\pi/8}\frac{1}{2}\sin(2x)\,dx\right]$$
$$= \frac{1}{A}\left[\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)\right]_0^{\pi/8}$$
$$= \frac{4}{\sqrt{2}}\left[\left(\frac{\pi}{16}\cdot\frac{\sqrt{2}}{2} + \frac{1}{4}\cdot\frac{\sqrt{2}}{2}\right) - \left(0 + \frac{1}{4}\right)\right]$$
$$= \boxed{\frac{\pi}{8} + \frac{1}{2} - \frac{1}{\sqrt{2}}.}$$

$$\bar{y} = \frac{1}{2A}\int_0^{\pi/8}(\cos(2x))^2\,dx$$
$$= \frac{2}{\sqrt{2}}\int_0^{\pi/8}\frac{1}{2}\left(1 + \cos(4x)\right)dx$$
$$= \frac{1}{\sqrt{2}}\left[x + \frac{1}{4}\sin(4x)\right]_0^{\pi/8}$$
$$= \frac{1}{\sqrt{2}}\left[\left(\frac{\pi}{8} + \frac{1}{4}(1)\right) - (0)\right]$$
$$= \boxed{\frac{\pi\sqrt{2}}{16} + \frac{\sqrt{2}}{8}.}$$

10. (Extra Credit 5 points) A tank is full of water. Find the work required to pump the water out of the spout. Use 9.8 m/s^2 for the acceleration due to gravity and assume that water has a denisty of 1000 kg/m^3. (Note: Significant partial credit will be awarded for correct set-up and stating the correct units of the final answer.)



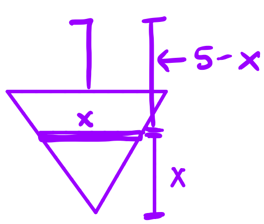


$\frac{w}{3-x} = \frac{3}{3} = 1$

$w = 3-x$

$V = L \cdot w \cdot h$
$= (8)(3-x)\Delta x$

$$W = \int_0^3 (2+x)(8)(3-x)(9.8)(1000)\,dx$$
$$= 78400 \int_0^3 (6+x-x^2)\,dx$$
$$= 78400\left[6x+\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^3$$
$$= 78400\left[18+\frac{9}{2}-9\right]$$
$$= 78400\left(\frac{27}{2}\right)$$
$$= 1{,}058{,}400 \text{ J.}$$



$W_i = F_i d_i$

$= 9.8\ m/s^2\ 1000\ kg/m^3\ \overbrace{x_i\ 8\ \Delta x\ m^3}^{\text{volume of slice}} \cdot (5-x_i)\ m$

$= 9.8(1000)(8)\ x_i\ \Delta x\ (5-x_i)$

$$\boxed{\begin{aligned} W &= \int_0^3 9.8(1000)\,8\,x(5-x)\,dx \\ &= 1{,}058{,}400\text{ J}\end{aligned}}$$