Your Name

Answer Key

Instructor Name (Ver Hoef or Zirbes)

Start Time

End Time

Page	Total Points	Score
2	16	
3	16	
4	16	
5	16	
6	20	
7	16	
8	(5 Extra Credit)	
Total	100	

- You will have 1 hour to complete the test.
- This test is closed notes and closed book and you may not use a calculator.
- Label any diagrams so as to indicate axes labels and scale.
- In order to receive full credit, you must **show your work.** Please write out your computations on the exam paper.
- Simplify all answers by fully distributing any constants.
- **PLACE A BOX AROUND** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question** where appropriate.

(16 points, 8 points each) Evaluate the following integrals.

Odd power, so pythagorean identity

1. $\int_0^{\pi/2} \sin^5 x dx = \int_0^{\pi/2} \sin^4 x \sin x \, dx$

$u = \cos x$
$du = -\sin x \, dx$
$x = 0 \rightarrow u = 1$
$x = \frac{\pi}{2} \rightarrow u = 0$

$= \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x \, dx$

$= \int_1^0 (1-u^2)^2 (-du)$

Note bounds!

$= \int_0^1 (1-u^2)^2 du$

$= \int_0^1 1 - 2u^2 + u^4 \, du$

$= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1$

$= \left[1 - \frac{2}{3}(1)^3 + \frac{1}{5}(1)^5\right] - \left[0 - \frac{1}{3}(0) + \frac{1}{5}(0)\right]$

$= 1 - \frac{2}{3} + \frac{1}{5} = \boxed{\frac{8}{15}}$ ✓

2. $\int \sin^{-1}(3x) dx$ IBP "Loner"

$\left[\begin{array}{ll} u = \sin^{-1}(3x) & dv = dx \\ du = \frac{3}{\sqrt{1-9x^2}} dx & v = x \end{array}\right.$

$= x \sin^{-1}(3x) - \int \frac{3x}{\sqrt{1-9x^2}} dx$

Now do a substitution.

$w = 1 - 9x^2 \quad dw = -18x \, dx$

$= x \sin^{-1}(3x) - \int \left(-\frac{3}{18}\right) \frac{1}{\sqrt{w}} dw$

$= x \sin^{-1}(3x) + \frac{1}{6} \int \frac{1}{\sqrt{w}} dw$

$= x \sin^{-1}(3x) + \frac{1}{6} \cdot 2\sqrt{w} + C$

$\boxed{= x \sin^{-1}(3x) + \frac{1}{3}\sqrt{1-9x^2} + C}$ ✓

(16 points, 8 points each) Evaluate the following integrals.

Even power, so half-angle identity.

3. $\int \cos^4(2\theta)d\theta$

$$= \int \left(\tfrac{1}{2}(1+\cos(4\theta))\right)^2 d\theta$$

$$= \frac{1}{4}\int 1 + 2\cos(4\theta) + \cos^2(4\theta)\, d\theta$$

$$= \frac{1}{4}\int 1 + 2\cos(4\theta) + \tfrac{1}{2}(1+\cos(8\theta))\, d\theta$$

$$= \frac{1}{4}\int \frac{3}{2} + 2\cos(4\theta) + \frac{1}{2}\cos(8\theta)\, d\theta$$

$$= \frac{1}{4}\left[\frac{3\theta}{2} + \frac{2}{4}\sin(4\theta) + \frac{1}{2}\frac{1}{8}\cos(8\theta)\right] + C$$

$$\boxed{= \frac{3\theta}{8} + \frac{1}{8}\sin(4\theta) + \frac{1}{64}\cos(8\theta) + C}$$

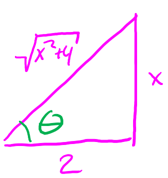
✓

4. $\int \frac{dx}{(x^2+4)^{3/2}}$ Trig sub w/ tangent

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\tan\theta = \frac{x}{2}$



$$= \int \frac{1}{(\sqrt{4\tan^2\theta+4})^3} 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{(\sqrt{4\sec^2\theta})^3} 2\sec^2\theta\, d\theta$$

$$= \int \frac{2\sec^2\theta}{2^3\sec^3\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta + C$$

$$\boxed{= \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C}$$

✓

(16 points, 8 points each) Evaluate the following integrals.

5. $\displaystyle\int e^x \cos(2x)\,dx$

IBP "Around-the-world"

$\left[\begin{array}{ll} u = \cos(2x) & dv = e^x\,dx \\ du = -2\sin(2x)\,dx & v = e^x \end{array}\right.$

$$\int e^x \cos(2x)\,dx = e^x\cos(2x) - \int e^x(-2\sin(2x))\,dx$$

$$= e^x\cos(2x) + 2\int e^x \sin(2x)\,dx$$

$\left[\begin{array}{ll} u = \sin(2x) & dv = e^x\,dx \\ du = 2\cos(2x)\,dx & v = e^x \end{array}\right.$

$$= e^x\cos(2x) + 2\left(e^x\sin(2x) - 2\int e^x\cos(2x)\,dx\right)$$

$$= e^x\cos(2x) + 2e^x\sin(2x) - 4\int e^x\cos(2x)\,dx$$

Let $I = \int e^x\cos(2x)\,dx$

$$I = e^x\cos(2x) + 2e^x\sin(2x) - 4I$$

$$5I = e^x\cos(2x) + 2e^x\sin(2x)$$

$$\boxed{\int e^x\cos(2x)\,dx = \frac{1}{5}e^x\cos(2x) + \frac{2}{5}e^x\sin(2x) + C}$$ ✓

Partial fractions, with Long division first

6. $\displaystyle\int \frac{x^3 - 4x - 10}{x^2 - x - 6}\,dx$

$$= \int x + 1 + \frac{3x+4}{(x-3)(x+2)}\,dx$$

$$\begin{array}{r} x + 1 \\ x^2 - x - 6\,\overline{)\,x^3 + 0x^2 - 4x - 10} \\ -(x^3 - x^2 - 6x) \\ \hline x^2 + 2x - 10 \\ -(x^2 - x - 6) \\ \hline 3x + 4 \end{array}$$

$$\frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$3x - 4 = A(x+2) + B(x-3)$$

$x = -2 \to -10 = B(-5) \to \boxed{B=2}$

$x = 3 \to 5 = A(5) \to \boxed{A=1}$

$$= \int x + 1 + \frac{1}{x-3} + \frac{2}{x+2}\,dx$$

$$\boxed{= \frac{1}{2}x^2 + x + \ln|x-3| + 2\ln|x+2| + C}$$ ✓

PFD irreducible Quadratic

(16 points, 8 points each) Evaluate the following integrals.

7. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}dx$ $= \displaystyle\int \frac{1}{x-1} + \frac{-x-1}{x^2+9}\,dx$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$10 = A(x^2+9)+(Bx+C)(x-1)$

$= Ax^2+9A+Bx^2+Cx-Bx-C$

$= (A+B)x^2+(C-B)x+(9A-C)$

$x=1 \rightarrow$

$10 = A(1+9)$

$10 = 10A$

$\boxed{A=1}$

$A+B=0 \quad C-B=0 \quad 9A-C=10$

$1+B=0 \quad C-(-1)=0$

$\boxed{B=-1} \quad \boxed{C=-1}$

$$= \int \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\,dx$$

$$\boxed{= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C}$$

✓

8. $\displaystyle\int_0^5 t\sqrt{25-t^2}\,dt$ $= \displaystyle\int_{25}^{0} \sqrt{u}\left(-\frac{1}{2}du\right)$

Could be $t = 5\sin\theta$, easier to do u-sub.

$u = 25-t^2$

$du = -2t\,dt$

$t=0 \rightarrow u=25$

$t=5 \rightarrow u=0$

$$= \frac{1}{2}\int_0^{25}\sqrt{u}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^{25}$$

$$= \frac{1}{3}(25)^{3/2} - 0$$

$$= \frac{1}{3}5^3$$

$$\boxed{= \frac{125}{3}}$$

✓

9. (4 points) Give the partial decomposition for the function $f(x) = \dfrac{2x+1}{(x+1)^3(x^2+4)^2}$ Do not determine the numerical value of the coefficeints.

$$\frac{2x+1}{(x+1)^3(x^2+4)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

(16 points, 8 points each) Determine whether each integral is convergent or divergent. Evaluate those that are convergent. Clearly explain why the integral diverges, if applicable.

10. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty} \int_e^b \frac{1}{x(\ln x)^3}\,dx$

$\left[u = \ln x \quad du = \frac{1}{x}dx \quad x = e \to u = 1 \quad x = b \to u = \ln b \right.$

$$= \lim_{b\to\infty} \int_1^{\ln b} \frac{1}{u^3}\,du$$

$$= \lim_{b\to\infty} \left[-\frac{1}{2u^2} \right]_1^{\ln b}$$

$$= \lim_{b\to\infty} \left[\frac{1}{-2(\ln b)^2} - \left(-\frac{1}{2}\right) \right] = \lim_{b\to\infty} \left[\frac{1}{2} - \frac{1}{2(\ln b)^2} \right]$$

$= \boxed{\frac{1}{2}}$, because $\lim_{b\to\infty} \ln b = \infty$. ✓

11. $\displaystyle\int_0^5 \frac{2}{(5-x)^4}\,dx$ Asymptote at $x=5$.

$$= \lim_{b\to 5^-} \int_0^b \frac{2}{(5-x)^4}\,dx = \lim_{b\to 5^-} \left[-\left(-\frac{2}{3(5-x)^3} \right) \right]_0^b$$

$$= \lim_{b\to 5^-} \frac{2}{3(5-x)^3} \Big|_0^b = \lim_{b\to 5^-} \frac{2}{3(5-b)^3} - \frac{2}{3(5)^3}$$

still has an asymptote.

$$= \infty$$

$\boxed{\text{Divergent.}}$

12. (16 points) Set up an expression that approximates the integral $\int_1^5 \frac{\cos x}{x}$ with $n = 4$ using the technique specified below.

(a) (4 points) Midpoint Rule. $\Delta x = \frac{5-1}{4} = 1$

$$\int_1^5 \frac{\cos x}{x}\,dx \approx 1\left[\frac{\cos(3/2)}{3/2} + \frac{\cos(5/2)}{5/2} + \frac{\cos(7/2)}{7/2} + \frac{\cos(9/2)}{9/2}\right]$$

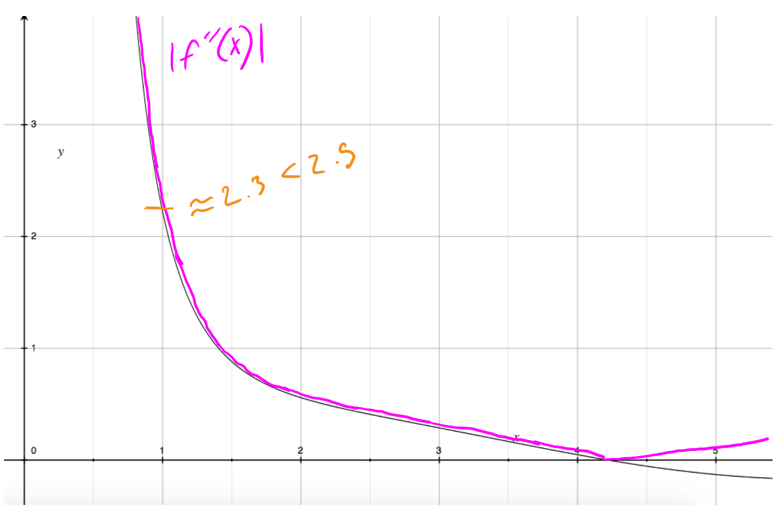
(b) (4 points) Trapezoid Rule.

$$\int_1^5 \frac{\cos x}{x}\,dx \approx \frac{1}{2}\left[\frac{\cos(1)}{1} + 2\,\frac{\cos(2)}{2} + 2\,\frac{\cos(3)}{3} + 2\,\frac{\cos(4)}{4} + \frac{\cos(5)}{5}\right]$$
$$= \frac{1}{2}\left(\cos(1) + \cos(2) + \frac{2}{3}\cos(3) + \frac{1}{2}\cos(4) + \frac{1}{5}\cos(5)\right)$$

(c) (4 points) Simpson's Rule.

$$\int_1^5 \frac{\cos x}{x}\,dx \approx \frac{1}{3}\left[\frac{\cos(1)}{(1)} + \frac{4\cos(2)}{2} + \frac{2\cos(3)}{3} + \frac{4\cos(4)}{4} + \frac{\cos(5)}{5}\right]$$
$$= \frac{1}{3}\left[\cos(1) + 2\cos(2) + \frac{2}{3}\cos(3) + \cos(4) + \frac{1}{5}\cos(5)\right]$$

(d) (4 points) Approximate the error, E_M, involved in the approximation from part (a) above using the graph of $f''(x)$ and error formula that are given below. Sketch the graph of $|f''(x)|$ on the same grid where $f''(x)$ is given and briefly explain how you chose the value of K.



I chose $K = 2.5$, being conservative to guarantee that $|f''(x)| < 2.5$ on $[1,5]$.

So,
$$|E_M| \le \frac{2.5(5-1)^3}{24(4)^2} \boxed{= 0.41\overline{6}}$$

Error Formula
$|E_M| \le \frac{K(b-a)^3}{24n^2}$ where $|f''(x)| \le K$ for $a \le x \le b$

13. (Extra Credit 5 points) Evaluate $\int \sin(\sqrt[3]{x})dx$

w-sub, then IBP twice

$w = \sqrt[3]{x}$

$dw = \frac{1}{3}x^{-2/3}dx$

$dx = 3x^{2/3}dw$

$= 3w^2 dw$

$$\int \sin(\sqrt[3]{x})dx = \int 3w^2 \sin w \, dw$$

$\begin{bmatrix} u = w^2 & dv = \sin w \, dw \\ du = 2w\,dw & v = -\cos w \end{bmatrix}$

$$= 3\left[-w^2\cos w + 2\int w\cos w \, dw\right]$$

$\begin{bmatrix} u = w & dv = \cos w \, dw \\ du = dw & v = \sin w \end{bmatrix}$

$$= 3\left[-w^2\cos w + 2\left[w\sin w - \int \sin w \, dw\right]\right]$$

$$= -3w^2\cos w + 6w\sin w + 6\cos w + C$$

$$\boxed{= -3\sqrt[3]{x^2}\cos\sqrt[3]{x} + 6\sqrt[3]{x}\sin\sqrt[3]{x} + 6\cos\sqrt[3]{x} + C}$$