Your Name

Start Time

End Time

Page	Total Points	Score
2	10	
3	10	
4	11	
5	13	
6	10	
7	10	
8	8	
9	10	
10	8	
11	10	
Total	100	

- You will have 2 hours to complete the test.
- This test is closed notes except for a 4 by 6 inch note card with hand written notes on both sides.
- No calculators allowed.
- Label any diagrams so as to indicate axes labels and scale.
- In order to receive full credit, you must **show your work.** Please write out your computations on the exam paper.
- When a problem asks you to **set up only** you do NOT need to simplify the integrand (the expression inside the integral sign) at all.
- When determining convergence or divergence of a series, state the test that is being applied. Common abbreviations are acceptable. **PLACE A BOX AROUND $\boxed{\textbf{YOUR FINAL ANSWER}}$ to each question** where appropriate.

1. (20 points) Evaluate the following integrals.

(a) $\int_0^{\pi/2} \sin^5 x dx$ $= \int_0^{\pi/2} \sin^4 x \sin x \, dx$

$= \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$x=0, \; u=1$

$x=\pi/2, \; u=0$

$= -\int_1^0 (1-u^2)^2 \, du$

$= \int_0^1 (1-2u^2+u^4) \, du$

$= \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) \Big|_0^1$

$= 1 - \frac{2}{3} + \frac{1}{5}$

$= \frac{15}{15} - \frac{10}{15} + \frac{3}{15}$

$= \boxed{\frac{8}{15}}$

(b) $\int \frac{2x^2-x+4}{x^3+x} dx$ $= \int \left(\frac{4}{x} + \frac{-2x-1}{x^2+1} \right) dx$

$\frac{2x^2-x+4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$2x^2-x+4 = A(x^2+1) + (Bx+C)x$

$2x^2-x+4 = Ax^2+A + Bx^2+Cx$

$= (A+B)x^2 + Cx + A$

So $A=4$, $C=-1$, $A+B=2$

$4+B=2$

$B=-2$

$= \int \left(\frac{4}{x} - \frac{2x}{x^2+1} - \frac{1}{x^2+1} \right) dx$

$= \boxed{4\ln|x| - \ln(x^2+1) - \tan^{-1} x + C}$

(c) $\int \sin^{-1}(5y)dy$ $= y\sin^{-1}(5y) - \int \frac{5y}{\sqrt{1-25y^2}}dy$

IBP

$u = \sin^{-1}(5y)$ $v = y$

$du = \frac{5}{\sqrt{1-25y^2}}dy$ $dv = dy$

$w = 1-25y^2$

$dw = -50y\,dy$

$$= y\sin^{-1}(5y) - \int \frac{5y}{\sqrt{w}}\left(\frac{-1}{50y}\right)dy$$

$$= y\sin^{-1}(5y) + \frac{1}{10}\int w^{-1/2}\,dw$$

$$= y\sin^{-1}(5y) + \frac{1}{10}\cdot\frac{2}{1}w^{1/2} + C$$

$$= \boxed{y\sin^{-1}(5y) + \frac{1}{5}\sqrt{1-25y^2} + C}$$

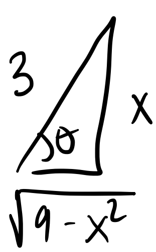
(d) $\int \frac{x^2}{\sqrt{(9-x^2)^3}}\,dx$ $= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{\left(\sqrt{9-9\sin^2\theta}\right)^3}$

Trig Sub

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$\frac{x}{3} = \sin\theta$



$$= \int \frac{27\sin^2\theta\cos\theta}{(3\cos\theta)^3}d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

$$= \int \tan^2\theta\, d\theta$$

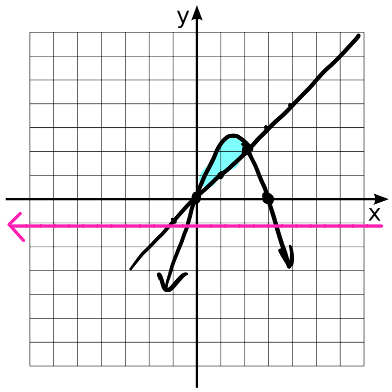
$$= \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C$$

$$= \boxed{\frac{x}{\sqrt{9-x^2}} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$

2. (11 points) Let R be the region bounded by the functions $f(x) = 3x - x^2$ and $g(x) = x$.

(a) (2 points) Graph the region and then set up, but do not solve, an integral that gives the **area** of R.



$3x - x^2 = x$

$2x - x^2 = 0$

$x(2-x) = 0$

$[x = 0, 2]$

$$A = \int_0^2 (3x - x^2 - x)\,dx$$

$$= \boxed{\int_0^2 (2x - x^2)\,dx}$$

(b) (3 points) Set up, but do not solve, an integral that finds the **volume** of the solid generated when R is rotated about the y-axis.

$r(x) = x$

$h(x) = 3x - x^2 - x$

$= 2x - x^2$

$$\boxed{\begin{aligned} V &= 2\pi \int_0^2 x(2x - x^2)\,dx \\ &= 2\pi \int_0^2 (2x^2 - x^3)\,dx \end{aligned}}$$

(c) (3 points) Set up, but do not solve, an integral that finds the volume of the solid generated when R is rotated about the line $y = -1$.

$r_{out} = 1 + 3x - x^2$

$r_{in} = 1 + x$

$$\boxed{V = \pi \int_0^2 \left((1 + 3x - x^2)^2 - (1+x)^2 \right) dx}$$

(d) (3 points) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Set up, but do not solve, an integral that gives the volume of this solid.

$$V = \int_0^2 (3x - x^2 - x)^2\,dx$$

$$= \boxed{\int_0^2 (2x - x^2)^2\,dx}$$

3. (4 points) Let $a_n = \tan^{-1}\left(\dfrac{n^2+1}{n^2+n+2}\right)$.

(a) Determine whether the sequence a_n converges or diverges. If it is convergent determine what it converges to.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan^{-1}\left(\frac{n^2+1}{n^2+n+2}\right)$$

$$= \tan^{-1}\left(\lim_{n\to\infty} \frac{1+1/n^2}{1+1/n+2/n^2}\right)$$

$$= \tan^{-1} 1$$

$$= \boxed{\pi/4}$$

$\boxed{a_n \text{ converges to } \pi/4}$

(b) Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. Justify your answer.

Since $\lim_{n\to\infty} a_n = \pi/4 \neq 0$, the series diverges by the nth term test/test for divergence.

4. (5 points) Find the sum of the following series exactly.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{5\cdot(-1)^n 2^n}{n!} = 5\cdot\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}$

$= 5\cdot e^{-2}$

$= \boxed{5/e^2}$

b) $\displaystyle\sum_{n=1}^{\infty} 3^{2n+1}10^{-n} = \sum_{n=1}^{\infty}\frac{3^{2n}\cdot 3}{10^n}$

$\displaystyle= \sum_{n=1}^{\infty} 3\left(\frac{9}{10}\right)^n$

$\displaystyle= \sum_{n=0}^{\infty} \frac{27}{10}\left(\frac{9}{10}\right)^n$

$\displaystyle= \frac{27/10}{1-9/10}$

$\displaystyle= \frac{27/10}{1/10}$

$= \boxed{27}$

5. (4 points) Find the Taylor series for the function $f(x) = 1/x$ centered at the point $a = -2$ using the definition. Give your answer in summation notation.

$f(x)=x^{-1}$
$f'(x)=-1x^{-2}$
$f''(x)=2x^{-3}$
$f'''(x)=-6x^{-4}$
$f''''(x)=24x^{-5}$

$$f(x) = f(-2) + f'(-2)(x+2) + \frac{f''(-2)}{2!}(x+2)^2 + \dots$$

$$= -\frac{1}{2} - \frac{1}{(-2)^2}(x+2) + \frac{1}{2!}\left(\frac{2}{(-2)^3}\right)(x+2)^2 + \frac{1}{3!}\frac{-6}{(-2)^4}(x+2)^3 + \dots$$

$$= -\frac{1}{2} - \frac{(x+2)}{2^2} - \frac{(x+2)^2}{2^3} - \frac{(x+2)^3}{2^4} - \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)(x+2)^n}{2^{n+1}}}$$

6. (8 points)

(a) (4 points) Determine whether the improper integral $\int_{2}^{\infty} \frac{\ln x}{x} dx$ converges or diverges. Evaluate it if it is convergent.

$$\boxed{\text{Thus the integral diverges.}}$$

$$\int_2^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_2^b \frac{\ln x}{x}\,dx$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u\,du$$

$u = \ln x$

$du = 1/x\,dx$

$x = 1,\ u = \ln 1 = 0$

$x = b,\ u = \ln b$

$$= \lim_{b\to\infty} \frac{1}{2}u^2 \Big|_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \left(\frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln 2)^2\right)$$

$$= \infty$$

(b) (4 points) Use the integral test, and your answer from (a), determine whether $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ converges or diverges. You must explicitly verify that the integral test applies to this series. No credit will be given if another test is used.

Let $f(x) = \frac{\ln x}{x}$, note that $f(x)$ is positive and continuous on $[2, \infty)$

To show f is decreasing we use the derivative

$$f'(x) = \frac{x \cdot 1/x - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2} < 0 \Rightarrow 1 - \ln x < 0$$

$$\Rightarrow 1 < \ln x$$

$$\Rightarrow x > e$$

Thus f is decreasing on (e, ∞).

As the improper integral diverges, so does the series.

7. (2 points) Determine the number of terms in the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ that we need to add to ensure $|\text{error}| \leq 0.001$.

$$|\text{error}| < b_{n+1} \leq 0.001$$

$$\frac{1}{(n+1)^3} \leq \frac{1}{1000}$$

$$1000 \leq (n+1)^3$$

$$10 \leq n+1$$

$$\boxed{n \geq 9}$$

8. (10 points) Determine whether the following series converge or diverge. You must clearly explain your reasoning and state any relavent tests by name.

(a) $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2+1}$

Note $0 \leq \left| \frac{\sin 2n}{n^2+1} \right| \leq \frac{1}{n^2+1} < \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} 1/n^2$ converges (p-series, $p=2$), $\sum_{n=1}^{\infty} \left| \frac{\sin 2n}{n^2+1} \right|$ also converges by DCT.

Thus the original series is absolutely convergent, and is therefore convergent.

(b) $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$

LCT with $a_n = \frac{5+2n}{1+2n^2+n^4}$, $b_n = \frac{n}{n^4} = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{5+2n}{1+2n^2+n^4} \right) \cdot \frac{n^3}{1}$$

$$= \lim_{n\to\infty} \frac{(5n^3+2n^4)\,1/n^4}{(1+2n^2+n^4)\,1/n^4}$$

$$= \lim_{n\to\infty} \frac{5/n + 2}{1/n^4 + 2/n^2 + 1}$$

$$= 2.$$

Since $\sum_{n=1}^{\infty} b_n$ converges (p-series, $p=3$), $\sum_{n=1}^{\infty} a_n$ also converges by LCT.

9. (8 points) Find the radius of convergence and the interval of convergence of the following series.

(2 pts) (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^n}$ Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2x+3)^n}{n^n}\right|}$$

$$= \lim_{n\to\infty} \frac{|2x+3|}{n}$$

$$= 0 < 1$$

So $\boxed{R=\infty, \quad \text{IOC } (-\infty,\infty)}$

(6 pts) (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n \cdot 5^n}$ Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+1)^{n+1}}{(n+1)\,5^{n+1}} \frac{n\,5^n}{(-1)^n(x+1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x+1|\,n}{5(n+1)}$$

$$= \frac{|x+1|}{5} < 1 \Rightarrow |x+1| < 5$$

$$\Rightarrow -5 < x+1 < 5$$

$$\Rightarrow -6 < x < 4$$

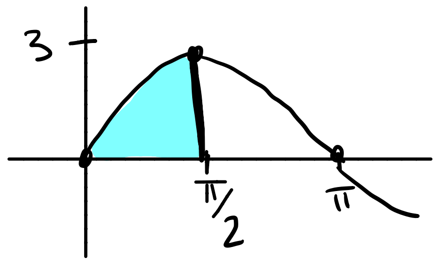
@ $x=-6 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-5)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ (diverges, harmonic)

@ $x=4 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,5^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (converges, AST)

IOC $\boxed{(-6,4] \quad \text{and} \quad R=5}$

10. (10 points) Let $\mathcal{R}$ be the region bounded by $y = 3\sin(x)$ and $y = 0$, $0 \le x \le \pi/2$.

(a) (2 point) Sketch the region and find the **AREA** of $\mathcal{R}$ here in PART A.



$$A = \int_0^{\pi/2} 3\sin x\,dx$$

$$= -3\cos x \Big|_0^{\pi/2}$$

$$= -3\cos\pi/2 + 3\cos 0$$

$$= \boxed{3}$$

(b) (8 points) Find the centroid of the the region $\mathcal{R}$.

$$\bar{x} = \frac{1}{A}\int_a^b x\,f(x)\,dx$$

$$= \frac{1}{3}\int_0^{\pi/2} x \cdot 3\sin x\,dx$$

$$= \int_0^{\pi/2} x\sin x\,dx$$

$$\begin{bmatrix} \text{IBP} & u = x & v = -\cos x \\ & du = dx & dv = \sin x\,dx \end{bmatrix}$$

$$= -x\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos x$$

$$= \left(-\frac{\pi}{2}\cdot 0 + 0\right) + \sin x\Big|_0^{\pi/2}$$

$$\boxed{\bar{x} = 1}$$

$$\bar{y} = \frac{1}{2A}\int_a^b (f(x))^2\,dx$$

$$= \frac{1}{6}\int_0^{\pi/2} 9\sin^2 x\,dx$$

$$= \frac{3}{2}\int_0^{\pi/2}\frac{1}{2}(1-\cos 2x)\,dx$$

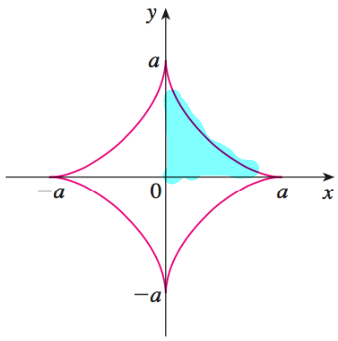
$$= \frac{3}{4}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi/2}$$

$$= \frac{3}{4}\left[\left(\pi/2 - \frac{1}{2}\sin\pi\right) - (0-0)\right]$$

$$\boxed{\bar{y} = \frac{3\pi}{8}}$$

11. (9 points) Consider the parametric curve where $x = a\sin^3\theta$ and $y = a\cos^3\theta$. A graph is given below in part (a).

(a) (2 points) Find and simplify $\frac{dy}{dx} = \frac{3a\cos^2\theta(-\sin\theta)}{3a\sin^2\theta\cos\theta}$



$= \boxed{-\frac{\cos\theta}{\sin\theta}}$

$= \boxed{-\cot\theta}$

for $0 \le \theta \le 2\pi$

(b) (2 points) Determine the location of any vertical tangents. Give the θ values only.

$\sin\theta = 0$

$\boxed{\theta = \pi/2, 3\pi/2}$

(c) (3 points) Set up, but do not solve, an integral that gives the area of the region.

$$A = 4\int_0^a y\,dx$$

$$= 4\int_0^{\pi/2} a\cos^3\theta \cdot 3a\sin^2\theta\cos\theta\,d\theta$$

$$= \boxed{12a^2\int_0^{\pi/2}\cos^4\theta\sin^2\theta\,d\theta}$$

(d) (3 points) Set up, but do not solve, an integral that gives the length of the curve.

$$L = 4\int_0^{\pi/2}\sqrt{(dy/d\theta)^2 + (dx/d\theta)^2}\,d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{(3a\cos^2\theta(-\sin\theta))^2 + (3a\sin^2\theta\cos\theta)^2}\,d\theta$$

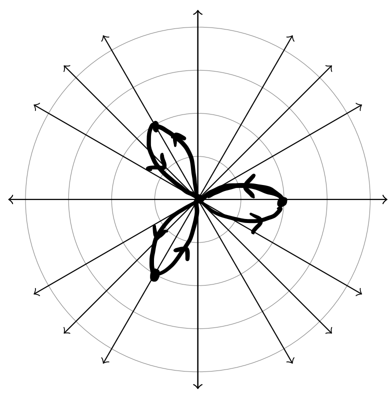
$$= 4\int_0^{\pi/2}\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

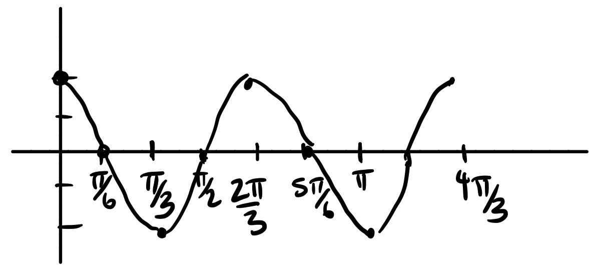
$$= 4\int_0^{\pi/2}\sqrt{9a^2\cos^2\theta\sin^2\theta(\sin^2\theta + \cos^2\theta)}\,d\theta$$

$$= 4\int_0^{\pi/2} 3a\cos\theta\sin\theta\,d\theta$$

12. (9 points) Consider the curve $r = \cancel{4\sin(3\theta)}$ $2\cos(3\theta)$.

(a) (3 points) Sketch the curve $r = 2\cos(3\theta)$.





(b) (4 points) Find the area enclosed by one petal.

$$A = 2 \cdot \int_0^{\pi/6} \tfrac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi/6} (2\cos 3\theta)^2 \, d\theta$$

$$= 4 \int_0^{\pi/6} \cos^2 3\theta \, d\theta$$

$$= 4 \cdot \tfrac{1}{2} \int_0^{\pi/6} (1 + \cos 6\theta) \, d\theta$$

$$= 2 \left(\theta + \tfrac{1}{6} \sin 6\theta \right) \Big|_0^{\pi/6}$$

$$= 2\left(\pi/6 + \tfrac{1}{6} \sin \pi - (0 + 0) \right)$$

$$= \boxed{\pi/3}$$

(c) (2 points) Set up, but do not solve, an integral that gives the length of the polar curve.

$$L = \int_0^{\pi} \sqrt{r^2 + (dr/d\theta)^2} \, d\theta \qquad dr/d\theta = -6\sin 3\theta$$

$$= \boxed{\int_0^{\pi} \sqrt{4\cos^2 3\theta + 36 \sin^2 3\theta} \, d\theta}$$