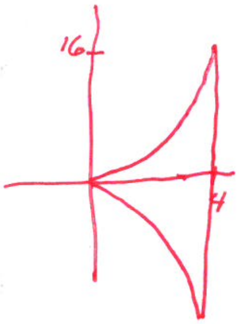
Math 252 Exam 2 Name: Key

10/19/2020 Instructor: Rhodes

Show all your work. Drawing pictures to help your reasoning is **strongly** encouraged.

1. (15 pts.) A metal plate of constant density is shaped like the region between the graphs of $y = x^2$ and $y = -x^2$ with $0 \leq x \leq 4$. Find the center of mass $(\bar{x}, \bar{y})$ of the plate. You may use the symmetry of the region to reduce the amount of calculation you need to do.



$\bar{y} = 0$ by symmetry

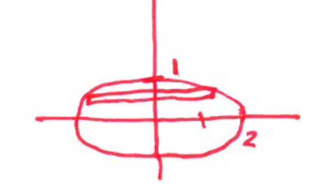
$$\bar{x} = \frac{\int_0^4 x(x^2-(-x^2))\,dx}{\int_0^4 (x^2-(-x^2))\,dx} = \frac{\int_0^4 2x^3\,dx}{\int_0^4 2x^2\,dx} = \frac{\frac{1}{2}x^4\Big|_0^4}{\frac{2}{3}x^3\Big|_0^4} = \frac{2\cdot 4^3}{\frac{2}{3}\cdot 4^3} = 3$$

$\bar{x} = 3$

2. (13 pts.) An elliptical window is to be located in a vertical wall of an underwater observatory. If the window has boundary given by the equation

$$x^2 + 4y^2 = 4$$

where x, y are measured in feet, and the surface of the water and is located at $y = 20$, find the force on the window due to the water. (The weight density of water is 62.4 lbs/ft^3.) Give your answer as an integral, but **do not evaluate** it. **Specify units** for your answer



$$F \approx \sum_{\text{strips}} (\text{area of strip})(\text{depth of strip})(62.4)$$

$$\approx \sum \left(2(2\sqrt{1-y^2})\right)\Delta y\,(20-y)(62.4)$$

$$F = \int_{-1}^{1} 62.4(20-y)\,4\sqrt{1-y^2}\,dy \quad \underline{\text{lbs}}$$

$x^2 + 4y^2 = 4$

$x^2 = 4 - 4y^2$

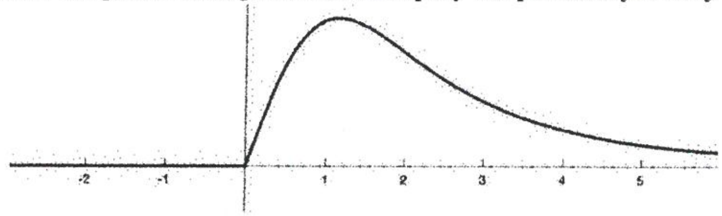
$x = \pm\sqrt{4-4y^2}$

$= \pm 2\sqrt{1-y^2}$

3. The length t, in minutes, of a telephone call placed through a certain company has probability density

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ A\dfrac{t}{(4+t^2)^2} & \text{for } t \geq 0 \end{cases}$$

where $A > 0$ is some constant, and the graph of f is shown.



(a) (6 pts.) Determine the value of A.

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} A\frac{t}{(4+t^2)^2}\,dt = A\int_4^{\infty} \frac{1}{2}\frac{1}{u^2}\,du$$

$u = 4+t^2$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$$= \frac{A}{2}\lim_{b\to\infty}\int_4^b \frac{1}{u^2}\,du = \frac{A}{2}\lim_{b\to\infty}\left(-\frac{1}{u}\Big|_4^b\right) = \frac{A}{2}\lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{4}\right)$$

So

$$1 = \frac{A}{2}\cdot\frac{1}{4} \quad \text{so } A = 8$$

(b) (8 pts.) Find the probability that a call lasts less than 3 minutes. (Use the value of A you found in part(a); if you did not find a value for A, you may assume $A = 7$.)

$$\int_0^3 8\frac{t}{(4+t^2)^2}\,dt = \frac{8}{2}\int_4^{13}\frac{1}{u^2}\,du = 4\left(-\frac{1}{u}\right)\Big|_4^{13}$$

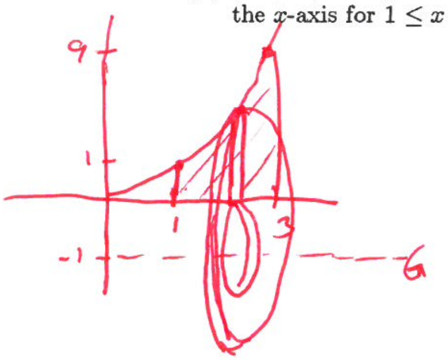
$u = 4+t^2$

$$= 4\left(-\frac{1}{13}+\frac{1}{4}\right) = 1 - \frac{4}{13} = \frac{9}{13}$$

(c) (4 pts.) Give an integral which computes the mean length of calls placed through the company. **Do not evaluate** your integral.

$$\int_{-\infty}^{\infty} t f(t)\,dt = \int_0^{\infty} t\, 8\frac{t}{(4+t^2)^2}\,dt$$

4. (a) (10 pts.) Compute the volume of the object obtained by rotating the region between $y = x^3$ and the x-axis for $1 \le x \le 3$ about the horizontal axis $y = -1$.



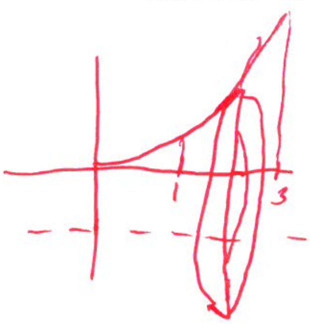
$$V \approx \sum_{\text{washers}} \left(\pi R^2 - \pi r^2\right)\Delta x$$

$$\approx \sum \left(\pi (x^3+1)^2 - \pi \cdot 1^2\right)\Delta x$$

$$V = \pi \int_1^3 \left((x^3+1)^2 - 1\right) dx = \pi \int_1^3 (x^6 + 2x^3)\, dx$$

$$= \pi\left(\frac{x^7}{7} + \frac{x^4}{2}\right)\Big|_1^3 = \pi\left(\frac{3^7}{7} + \frac{3^4}{2} - \frac{1}{7} - \frac{1}{2}\right) = \pi\left(\frac{2186}{7} + 40\right)$$

(b) (10 pts.) Give an integral to compute the surface area of the object obtained by rotating the graph of $y = x^3$, $1 \le x \le 3$ about the axis $y = -1$. **Do not evaluate** your integral.

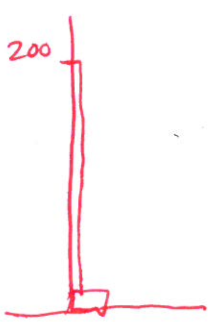


$$A \approx \sum_{\substack{\text{pieces}\\ \text{of}\\ \text{cones}}} 2\pi R\, \Delta s \approx \sum 2\pi (x^3+1)\sqrt{1 + (3x^2)^2}\, \Delta x$$

$f'(x)$ for $f(x) = x^3$

$$A = \int_1^3 2\pi (x^3+1)\sqrt{1 + 9x^4}\, dx$$

5. (13 pts.) A 180 lb person is attached to the bottom of a rope that hangs 200 ft over a cliff. The rope weights 0.25 lbs/ft. How much work will be done in pulling the person and rope up to the top of the cliff? **Specify units** for your answer.



$$W \approx \text{Work for person} + \sum_{\substack{\text{piece}\\ \text{of}\\ \text{rope}}} (200 - y)\,.25\,\Delta y$$

$$W = 180 \cdot 200 + \frac{1}{4}\int_0^{200} (200 - y)\, dy$$

$$= 36{,}000 + \frac{1}{4}\left(200y - \frac{y^2}{2}\right)\Big|_0^{200}$$

$$= 36{,}000 + \frac{1}{4}(40{,}000 - 20{,}000)$$

$$= 41{,}000 \ \text{ft} \cdot \text{lbs}$$

6. (8 pts.) The temperature, in $°C$, over a 24 hour period is given by the function

$$f(t) = 18 + t/3 + 3\sin(\pi t/12), \quad 0 \le t \le 24.$$

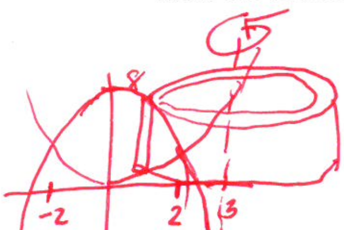
What is the average temperature over that time? **Specify units** for your answer.

$$AV = \frac{\int_0^{24} 18 + t/3 + 3\sin(\pi t/12)\, dt}{24} = \frac{\left(18t + \frac{t^2}{6} - \frac{36}{\pi}\cos\left(\frac{\pi t}{12}\right)\right)\Big|_0^{24}}{24}$$

$$= \frac{18 \cdot 24 + 24^2/6 - \frac{36}{\pi}\cos(2\pi) - \left(0 + 0 - \frac{36}{\pi}\cos 0\right)}{24}$$

$$= 18 + \frac{24}{6} + \frac{36}{24\pi}\left(\underbrace{\cos 2\pi}_{=1} - \underbrace{\cos 0}_{=1}\right) = 18 + 4 = 22°C$$

7. (13 pts.) Find the volume of the object obtained by rotating the region between $y = 8 - x^2$ and $y = x^2$ about the vertical axis $x = 3$.



$8 - x^2 = x^2$

$8 = 2x^2$

$x = \pm 2$

$$V \approx \sum 2\pi R \cdot \text{height} \cdot \Delta x$$

cylindrical shells

$$\approx \sum 2\pi(3-x)\left((8-x^2) - x^2\right)\Delta x$$

$$V = 2\pi \int_{-2}^{2} (3-x)(8-2x^2)\, dx$$

$$= 2\pi \int_{-2}^{2} (24 - 8x - 6x^2 + 2x^3)\, dx$$

$$= 2\pi\left(24x - 4x^2 - 2x^3 + \tfrac{1}{2}x^4\right)\Big|_{-2}^{2}$$

$$= 2\pi\left((48 - \cancel{16} - 16 + \cancel{8}) - (-48 - \cancel{16} + 16 + \cancel{8})\right)$$

$$= 4\pi(32) = 128\pi$$