Math 252 Exam 1
09/21/2020

Name: Key
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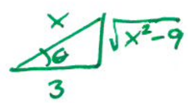
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$	$\sin 2x = 2\sin x\cos x$
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$	$\cos 2x = \cos^2 x - \sin^2 x$

1. (65 pts. – 13 pts. each) Evaluate the following indefinite integrals, showing your work.

(a) $\int \frac{1}{(x^2-9)^{3/2}}\,dx \quad = \int \left(\frac{1}{3}\cot\theta\right)^3 3\sec\theta\tan\theta\,d\theta = \frac{1}{9}\int \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\cos\theta}\,d\theta$



$\frac{3}{x} = \cos\theta \Rightarrow x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\,d\theta$

$\frac{3}{\sqrt{x^2-9}} = \cot\theta \Rightarrow \frac{1}{\sqrt{x^2+9}} = \frac{1}{3}\cot\theta$

$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{1}{u^2}\,du = -\frac{1}{9}\frac{1}{u} + C = -\frac{1}{9}\frac{1}{\sin\theta} + C$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$= -\frac{1}{9}\frac{1}{\frac{\sqrt{x^2-9}}{x}} + C = -\frac{1}{9}\frac{x}{\sqrt{x^2-9}} + C$

(b) $\int_0^1 \arctan x\,dx \quad = x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx = x\arctan x\Big|_0^1 - \frac{1}{2}\int_1^2 \frac{dw}{w}$

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x$

$w = 1+x^2$

$dw = 2x\,dx$

$= x\arctan x\Big|_0^1 - \frac{1}{2}\ln w\Big|_1^2 = (\arctan 1 - 0) - \frac{1}{2}(\ln 2 - \ln 1)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

(c) $\int \sec^3 2x \tan^3 2x \, dx$ $= \int \sec^2 2x \tan^2 2x \, (\sec 2x \tan 2x) \, dx$

$$= \int \sec^2 2x \, (\sec^2 2x - 1) \, (\sec 2x \tan 2x) \, dx$$

$$u = \sec 2x$$

$$du = 2 \sec 2x \tan 2x \, dx$$

$$= \frac{1}{2} \int u^2 (u^2 - 1) \, du = \frac{1}{2} \int (u^4 - u^2) \, du$$

$$= \frac{1}{2} \left(\frac{u^5}{5} - \frac{u^3}{3} \right) + C$$

$$= \frac{1}{10} \sec^5 2x - \frac{1}{6} \sec^3 2x + C$$

(d) $\int \frac{2x^2 - 2x - 2}{x^3 - x} \, dx$

$$\frac{2x^2 - 2x - 2}{x^3 - x} = \frac{2x^2 - 2x - 2}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$2x^2 - 2x - 2 = A(x+1)(x-1) + Bx(x-1) + C(x)(x+1)$$

$\underline{x = 0}$ $\quad -2 = -A \Rightarrow A = 2$

$\underline{x = 1}$ $\quad -2 = 2C \Rightarrow C = -1$

$\underline{x = -1}$ $\quad 2 = 2B \Rightarrow B = 1$

so the integral $= \int \frac{2}{x} + \frac{1}{x+1} - \frac{1}{x-1} \, dx$

$$= 2 \ln|x| + \ln|x+1| \cancel{+} - \ln|x-1| + C$$

(e) $\int_3^7 x\sqrt{x-3}\,dx \quad = \int_0^4 (u+3)\sqrt{u}\,du = \int_0^4 u^{3/2} + 3u^{1/2}\,du$

$u = x-3 \qquad x = u+3$

$du = dx$

$= \frac{2}{5}u^{5/2} + 2u^{3/2}\Big|_0^4 = \frac{2}{5}(4^{5/2}) + 2(4^{3/2})$

$= \frac{2}{5}2^5 + 2\cdot 2^3 = \frac{64}{5} + 16 = \frac{144}{5}$

2. (20 pts. – 10 pts. each) Determine whether the following improper integrals converge or diverge. If they converge, compute their values.

(a) $\int_0^\infty \frac{1}{(1+x)^{5/2}}\,dx \quad = \lim_{b\to\infty} \int_0^b \frac{1}{(1+x)^{5/2}}\,dx = \lim_{b\to\infty} \left(\frac{-2}{3}(1+x)^{-3/2}\Big|_0^b \right)$

$= \lim_{b\to\infty} \left(-\frac{2}{3}\frac{1}{(1+b)^{3/2}} + \frac{2}{3} \right)$

$= 0 + \frac{2}{3} = \frac{2}{3}$ converges

(b) $\int_0^{\pi/2} \tan\theta\,d\theta \quad = \lim_{b\to\frac{\pi}{2}^-} \int_0^b \tan\theta\,d\theta$

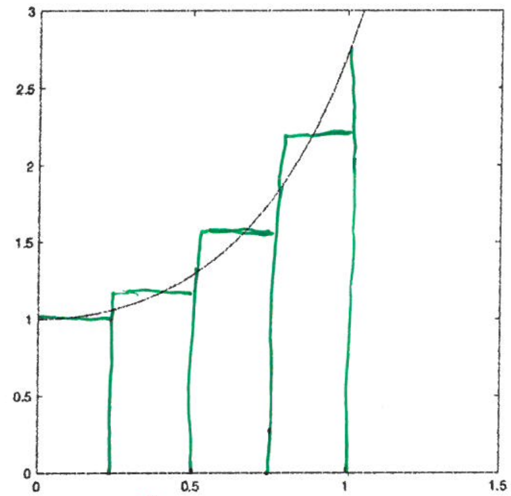
$= \lim_{b\to\frac{\pi}{2}^-} \int_0^b \frac{\sin\theta}{\cos\theta}\,d\theta$

$= \lim_{b\to\frac{\pi}{2}^-} \left(-\ln|\cos\theta| \Big|_0^b \right)$

$= \lim_{b\to\frac{\pi}{2}^-} \left(-\ln|\cos b| - \ln 1 \right)$

$= \lim_{b\to\frac{\pi}{2}^-} \left(-\ln|\underbrace{\cos b}_{\to 0}| \right) = +\infty$ diverges

3. (15 pts.) The function $f(x) = e^{(x^2)}$ is graphed.



(a) (2 pts.) On the graph, sketch the rectangles whose areas would form the midpoint sum M_4 approximating the integral $\int_0^1 e^{(x^2)}\,dx$.

(b) (5 pts.) Write out the midpoint sum M_4, using numbers in all terms. **Do not simplify.**

$$\frac{1}{4}\left(e^{\left(\frac{1}{8}\right)^2} + e^{\left(\frac{3}{8}\right)^2} + e^{\left(\frac{5}{8}\right)^2} + e^{\left(\frac{7}{8}\right)^2}\right)$$

(c) (3 pts.) Does M_4 give too large or too small an estimate? Briefly explain how you can tell.

Too small. The function is concave up, + pivoting the tops of the rectangles to become tangent at the midpoint shows the area they give is below the true area.

(d) (5 pts.) The error bound for a midpoint sum is

$$|\text{Error}(M_n)| \le \frac{K(b-a)^3}{24n^2},$$

where $|f''(x)| \le K$ on $[a, b]$. Use this to determine n so that the error in approximating $\int_0^1 e^{(x^2)}\,dx$ is no larger than 0.0001.

$f'(x) = 2x e^{x^2}$

$f''(x) = 2e^{x^2} + (2x)^2 e^{x^2} = (2+4x^2)e^{x^2}$

on $0 \le x \le 1$ $f''(x)$ is largest at $x = 1$, $f''(x) = 6e$

So we take $K = 6e$

Thus we want $\frac{6e(1-0)^3}{24n^2} \le 10^{-4}$

$\frac{e}{4n^2} < 10^{-4}$

$\frac{e}{4} \cdot 10^4 < n^2$

$\sqrt{\frac{e}{4} 10^4} < n$

i.e. $n > \sqrt{e} \cdot \frac{100}{2} = 50\sqrt{e}$